

# Functional renormalization group for d-wave superconductivity in Hubbard type models

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The temperature dependence of d-wave superconducting order for two dimensional fermions with d-wave attraction is investigated by means of the functional renormalization group with partial bosonization. Below the critical temperature  $T_c$  we find superconductivity, a gap in the electron propagator and a temperature dependent anomalous dimension. At  $T_c$  the renormalized “superfluid density” jumps and the approach to  $T_c$  from above is characterized by essential scaling. These features are characteristic for a phase transition of the Kosterlitz-Thouless (KT) type.

The two-dimensional Hubbard model [1] is often associated with high temperature superconductivity [2]. Its ground state is assumed to be a d-wave superconductor for an appropriate range of doping [3, 4]. Under the hypothesis of d-wave superconductivity we study within a simplified effective model the phase transition from the high temperature “symmetric phase” to the low temperature superconducting phase by means of the functional renormalization group. As characteristic features we find (i) essential scaling as the critical temperature  $T_c$  is approached from above, (ii) a massless Goldstone mode for  $T < T_c$  leading to superfluidity, (iii) a temperature dependent anomalous dimension  $\eta$  for  $T < T_c$ , (iv) a jump in the renormalized superfluid density at  $T_c$  and (v) a gap in the electron propagator for  $T < T_c$  which vanishes non-analytically as  $T \rightarrow T_c$ .

The features (i)-(iv) are characteristic for a KT phase transition [5] which is a natural candidate for a universality class with a global  $U(1)$  symmetry. Actually, while in the infinite volume limit the “bare” order parameter vanishes in accordance with the Mermin-Wagner theorem, a renormalized order parameter [6] spontaneously breaks the  $U(1)$  symmetry for  $T < T_c$ , leading to an infinite correlation length for the Goldstone mode. Vortices arise as topological defects for  $T < T_c$ . In the Hubbard model the fermionic excitations (single electrons or holes) are an important ingredient for driving the transition. Finally, coupling our model to electromagnetic fields the photon acquires a mass through the Higgs mechanism while the Goldstone mode disappears from the spectrum, being a gauge degree of freedom. Superfluidity is replaced by superconductivity. Recently, various experimental data have been interpreted as signatures of KT phase fluctuations [7, 8, 9, 10]. The direct relation between real high temperature superconductors and the two-dimensional Hubbard model is difficult, however.

The problem of d-wave superconductivity in the Hubbard model can be separated into two qualitative steps in the renormalization flow. In the first step the fluctuations with momenta  $|\mathbf{p}^2 - \mathbf{p}_F^2| > \Lambda^2$  (with  $\mathbf{p}_F$  on the Fermi-surface) have to generate a strong effective interaction in the d-wave channel. This phenomenon has indeed been found in renormalization group studies [11, 12, 13, 14, 15, 16, 17, 18, 19], the d-wave coupling being triggered by spin wave fluctuations. It also has been found in various other approaches, see e.g. [20, 21, 22]. The second step, involving momenta  $|\mathbf{p}^2 - \mathbf{p}_F^2| < \Lambda^2$  for the fermions and  $\mathbf{p}^2 < \Lambda^2$  for bosonic bound states, has to deal with a situation where the d-wave channel coupling domi-

nates. We only address here the second step. We find that the universal behavior in the vicinity of the critical temperature is dominated by effective bosonic fluctuations and becomes independent of the microscopic details. This justifies our approach with a single coupling  $\lambda_d$  at the scale  $\Lambda$  in the d-wave channel, even though this remains, of course, only an approximation. The prize to pay is that  $\lambda_d$  is only an effective coupling – its relation to the parameters  $t$  and  $U$  of the Hubbard model is expected to depend on  $T$  and the chemical potential  $\mu$ .

We emphasize that we obtain a KT-like phase transition within a purely fermionic model. The effective bosonic degrees of freedom arise as composite fields – we do not start from a bosonic effective theory. The transition from a fermionic to a bosonic description is a result of the renormalization flow. The complete decoupling of the fermionic degrees of freedom for the description of the critical behavior is a non-trivial result which only holds for a sufficiently large critical temperature  $T_c$ . (For  $T_c \rightarrow 0$  interesting modifications reflect the different universality class of a quantum phase transition.) Several new features are directly related to the dominance of fermionic fluctuations sufficiently away from the critical region: We find that the critical temperature  $T_c$  is below a “pseudo-critical temperature”  $T_{pc}$  where the effective interaction strength for the electrons diverges. On the other hand, the temperature range  $T_c < T < T_{pc}$  is strongly affected by fluctuations of composite bosons. For the transition to d-wave superconductivity we find that  $T_c$  is actually rather near to  $T_{pc}$ . This contrasts to the transition to antiferromagnetism for  $\mu = 0$  where  $T_c$  has been found substantially below  $T_{pc}$  [23, 24]. Other fermionic aspects concern the relation between the gap in the electron propagator and the superfluid order. We also find a quantitatively important finite size effect which modifies the essential scaling for  $T \rightarrow T_c$ . Again it is dominated by the fermionic fluctuations.

Our starting point is an action with nearest and next-nearest neighbor hopping on a two dimensional square lattice with an attractive interaction in the d-wave channel

$$\begin{aligned} \Gamma_\Lambda^F[\psi] = & \sum_Q \psi^\dagger(Q)(i\omega_F + \epsilon_{\mathbf{q}} - \mu)\psi(Q) \\ & - \lambda_d \sum_{K_1, \dots, K_4} \delta_{K_1+K_2, K_3+K_4} d((\mathbf{k}_1 - \mathbf{k}_2)/2) \\ & \times d((\mathbf{k}_3 - \mathbf{k}_4)/2) \psi_\downarrow^*(K_3) \psi_\uparrow^*(K_4) \psi_\uparrow(K_2) \psi_\downarrow(K_1). \end{aligned} \quad (1)$$

We employ the short-hand notation  $Q = (\omega_F, \mathbf{q})$ ,  $\omega_F = (2n +$

$1)\pi T, n \in \mathbb{Z}, \sum_Q = T \sum_n \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2}$ . The Fermi surface is specified by  $\epsilon_{\mathbf{q}} = -2t(\cos q_1 + \cos q_2) - 4t' \cos q_1 \cos q_2 = \mu$  and  $d(\mathbf{q}) = \cos q_1 - \cos q_2$  is a d-wave form factor. All lengths are measured in units of the lattice distance which we set to unity. We use partial bosonization with a bosonic field  $\phi$  standing for a fermion bilinear  $\sim \psi\psi$  in the d-wave channel. For the integration of the fluctuations we employ the concept of the effective average action  $\Gamma_k$  [25], which amounts to the quantum effective action (generating 1PI-vertices) in presence of an infrared cutoff  $k$ . It interpolates between the microscopic action  $\Gamma_{k \rightarrow \infty} \approx S$  and the full quantum effective action  $\Gamma_{k \rightarrow 0} = \Gamma$ .

The flow of the effective average action obeys an exact renormalization group equation [26],

$$\partial_k \Gamma_k[\phi, \psi] = \frac{1}{2} \text{STr} \left\{ \tilde{\partial}_k \ln [\Gamma_k^{(2)}[\phi, \psi] + R_k] \right\}, \quad (2)$$

where  $R_k$  denotes an infrared cutoff and  $\tilde{\partial}_k = \sum_i (\partial_k R_k^i) \frac{\partial}{\partial R_k^i}$  (the index  $i$  refers to cutoffs for fermions and bosons). The “supertrace”  $\text{STr}$  runs over field type, momentum and internal indices, and has an additional minus sign for fermionic entries. The functional differential equation (2) involves the logarithm of the full inverse propagator  $\Gamma_k^{(2)}[\phi, \psi]$  in presence of background fields (second functional derivative of  $\Gamma_k$ ), regulated by  $R_k$ . Approximations to the solution of eq. (2) proceed by a truncation of  $\Gamma_k$  on the r.h.s. with a suitable ansatz.

Our truncation for the effective average action is

$$\begin{aligned} \Gamma_k = & \sum_Q \psi^\dagger(Q) (i\omega_F + \epsilon_{\mathbf{q}} - \mu) \psi(Q) \\ & + \sum_Q \phi^*(Q) (iZ\omega_B + A[\mathbf{q}]^2 t^2) \phi(Q) + \sum_X U(\phi) \\ & - \sum_{K, Q, Q'} \delta_{K, Q+Q'} d((\mathbf{q} - \mathbf{q}')/2) \\ & \times (\phi^*(K) \psi_\uparrow(Q) \psi_\downarrow(Q') - \phi(K) \psi_\uparrow^*(Q) \psi_\downarrow^*(Q')), \end{aligned} \quad (3)$$

with  $X = (\tau, \mathbf{x})$ ,  $\sum_X = \int_0^1 d\tau \sum_{\mathbf{x}}$  and for bosons  $Q = (\omega_B, \mathbf{q})$ ,  $\omega_B = 2m\pi T$ ,  $m \in \mathbb{Z}$ . Further  $[\mathbf{q}]^2$  is defined as  $[\mathbf{q}]^2 = q_1^2 + q_2^2$  for  $q_i \in [-\pi, \pi]$  and continued periodically otherwise. We approximate the effective potential by a quartic polynomial

$$U(\phi) = \begin{cases} \bar{m}^2 \delta + \frac{\bar{\lambda}_\phi}{2} \delta^2 & (\text{SYM}) \\ \frac{\bar{\lambda}_\phi}{2} (\delta - \delta_0)^2 & (\text{SSB}) \end{cases}, \quad (4)$$

with  $\delta = \phi^* \phi$ . Here (SYM) is used as long as the minimum of  $U$  is located at  $\phi = 0$ , while the regime with spontaneous symmetry breaking (SSB) is characterized by non vanishing  $\delta_0 = \phi_0^* \phi_0$ . The order parameter  $\phi_0$  determines the size of the gap in the fermion propagator  $\Delta(\mathbf{q}) = |d(\mathbf{q}) \phi_0|$ . The generalized couplings  $Z$ ,  $A$ ,  $\bar{m}^2$ ,  $\delta_0$  and  $\bar{\lambda}_\phi$  depend on  $k$ . At the scale  $\Lambda$  our truncation is equivalent to the fermionic action (1) provided

$$(\bar{m}/t)^2 = \frac{1}{\lambda_d/t}, \quad \bar{\lambda}_\phi = 0, \quad Z = A = 0, \quad (5)$$

as can be seen by solving for  $\phi$  as a functional of  $\psi$ .

We will follow the flow until a physical scale  $k_{ph}$  where  $k_{ph}^{-1}$  corresponds to the macroscopic size of the experimental probe. (In practice, our choice  $k_{ph}/t = 10^{-9} = e^{-20.723}$  corresponds to a probe size of roughly 1 cm.) For  $T < T_c$  we find a non vanishing superfluid density  $\delta_0(k_{ph})$  and therefore a non vanishing gap  $\Delta$ , while  $\delta_0(k_{ph} \rightarrow 0) = 0$ .

In addition to the truncation we have to specify the regulator functions  $R_k$ . For the fermion fluctuations we use a temperature like regulator,

$$R_k^F(Q) = i\omega_F \cdot \left( \frac{T_k}{T} - 1 \right), \quad \text{with} \quad T_k^4 = T^4 + k^4. \quad (6)$$

For  $k \gtrsim T$  the cutoff  $R_k^F$  in the inverse fermion propagator suppresses the contribution of all fluctuations with momenta  $|\mathbf{p} - \mathbf{p}_F|^2 < (\pi k)^2$ , even for  $T = 0$ . It becomes ineffective for  $k \ll T$  where no cutoff is needed anymore. The bosonic cutoff regularizes the fluctuations of long range bosonic modes. We use a “linear cutoff” for the space like momenta [27, 28]

$$R_k^B(Q) = A \cdot (k^2 - [\mathbf{q}]^2 t^2) \Theta(k^2 - [\mathbf{q}]^2 t^2). \quad (7)$$

The flow equations for the couplings are obtained by projection from (2). In (SYM) the effective potential flows as

$$\begin{aligned} \partial_k U_k(\delta) = & \frac{2T(\partial_k T_k)}{T_k} \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} \gamma_\delta \tanh \gamma_\delta \\ & + \sum_Q \frac{\partial_k R_k^\phi(Q) [P_k^B(-Q) + \bar{m}^2 + 2\bar{\lambda}_\phi \delta]}{[P_k^B(Q) + \bar{m}^2 + 2\bar{\lambda}_\phi \delta] [P_k^B(-Q) + \bar{m}^2 + 2\bar{\lambda}_\phi \delta] - \bar{\lambda}_\phi^2 \delta^2}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \gamma_\delta = & \frac{1}{2T_k} \sqrt{(\epsilon_{\mathbf{q}} - \mu)^2 + \Delta^2(\mathbf{q})}, \\ P_k^B(Q) = & iZ\omega_B + A[\mathbf{q}]^2 t^2 + R_k^B(Q). \end{aligned} \quad (9)$$

For the mass parameter and the bosonic quartic coupling this yields

$$\partial_k \bar{m}^2 = \frac{\partial}{\partial \delta} \partial_k U_k(\delta)|_{\delta=0}, \quad \partial_k \bar{\lambda}_\phi = \frac{\partial^2}{\partial \delta^2} \partial_k U_k(\delta)|_{\delta=0}. \quad (10)$$

The fermion fluctuations generate kinetic and gradient terms for the bound state bosons (SYM regime)

$$\begin{aligned} \partial_k Z = & -T(\partial_k T_k) \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} d(\mathbf{q})^2 \frac{\partial}{\partial T_k} \frac{\partial}{\partial i\omega} \\ & \times \frac{\tanh \frac{\epsilon_{\mathbf{q}} - \mu}{2T_k} + \tanh(\frac{\epsilon_{\mathbf{q}} - \mu}{2T_k} + \frac{i\omega}{2T})}{T_k^2 (\frac{\epsilon_{\mathbf{q}} - \mu}{T_k} + \frac{i\omega}{2T})} \Big|_{\omega=0}, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \partial_k A = & -2T(\partial_k T_k) \frac{1}{t^2} \frac{\partial}{\partial T_k} \frac{\partial}{\partial (l^2)} \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} d(\mathbf{q} - \frac{l}{2} \mathbf{e}_1)^2 \\ & \times \frac{\tanh(\frac{\epsilon_{\mathbf{q}} - \mu}{2T_k}) + \tanh(\frac{\epsilon_{\mathbf{e}_1 - \mathbf{q}} - \mu}{2T_k})}{T_k (\epsilon_{\mathbf{q}} + \epsilon_{\mathbf{e}_1 - \mathbf{q}} - 2\mu)} \Big|_{l=0}. \end{aligned} \quad (12)$$

There are no bosonic contributions to the flow of  $Z$  and  $A$  to this level of approximation. The flow starts in the symmetric

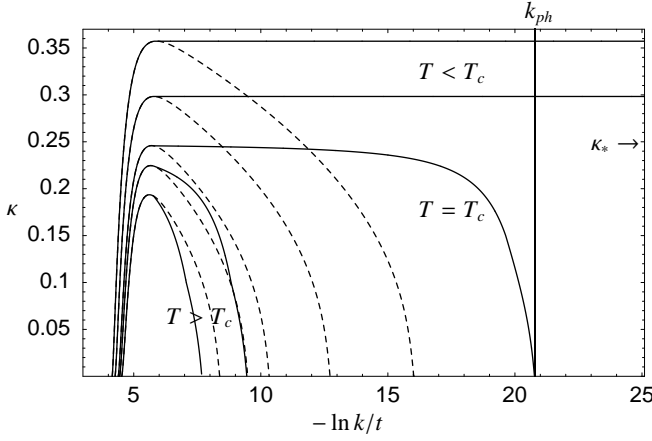


FIG. 1: Flow of  $\kappa(k)$  for  $T/t = (0.037, 0.0369, T_c/t, 0.0365, 0.036)$  with  $T_c/t = 0.03681$  and  $t'/t = -0.1$ ,  $\lambda_d/t = 5/8$ ,  $\mu/t = -0.6$ .

regime (4) with initial values (5) and we choose  $\Lambda/t = 0.5$ . For temperatures below a pseudo-critical temperature  $T_{pc}$  the scale dependent mass term  $\bar{m}^2$  vanishes at a certain critical scale  $k_{SSB}$ . This signals local ordering and indeed corresponds to a divergent effective four fermion coupling  $\bar{\lambda}_d(k)$ . Below  $k_{SSB}$  we use the truncation (SSB) in (4).

We do not consider here the region of very small  $T$  where interesting quantum critical phenomena may occur [29]. Then classical statistics dominates the flow for  $k \ll T$ . A typical flow in the SSB regime is depicted in fig. 1, where we show  $\kappa = t^2 A \delta_0 / T$  for various values of  $T$  and  $t'/t = -0.1$ ,  $\lambda_d/t = 5/8$ ,  $\mu/t = -0.6$ . The fermion fluctuations (first term in eq. (8)) first drive  $\bar{m}^2(k)$  to zero and induce non vanishing  $\kappa(k)$ , starting from  $\kappa(k_{SSB}) = 0$ . For  $k \ll k_{SSB}$  the effect of the fermion fluctuations is strongly reduced due to the suppression with powers  $k/T$  from  $\partial_k T_k$  and the presence of a non vanishing gap  $\Delta$ . Now the bosonic fluctuations take over and tend to reduce the value of  $\kappa(k)$  as  $k$  is lowered further. For the region around and on the right of the maximum of  $\kappa(k)$  in fig. 1 we have already  $k \ll T$ . Then the Matsubara frequency  $m = 0$  dominates the bosonic contributions and the system gets reduced to classical statistics for the  $O(2)$  linear  $\sigma$ -model. One can see this effective reduction from 2 + 1 to 2 dimensions explicitly by evaluating the bosonic contribution in eq. (8) (the second term, denoted by  $\partial_k U_k^B$ ). Keeping only the  $m = 0$  contribution the flow equation simplifies considerably

$$\partial_k U_k^B(\delta) = \frac{TAk}{2} \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} (2 - \eta(k)(1 - \frac{\mathbf{q}^2 t^2}{k^2})) \Theta(k^2 - \mathbf{q}^2 t^2) \times \left\{ \frac{1}{Ak^2 + (\delta - \delta_0)\bar{\lambda}_\phi} + \frac{1}{Ak^2 + (3\delta - \delta_0)\bar{\lambda}_\phi} \right\}. \quad (13)$$

Here  $\eta(k) = -\frac{\partial \ln A}{\partial \ln k}$  defines the anomalous dimension.

For a better understanding of the scaling behavior we introduce rescaled and renormalized quantities

$$u = \frac{t^2}{Tk^2} U, \quad \tilde{\delta} = \frac{t^2 A}{T} \delta, \quad \kappa = \frac{t^2 A}{T} \delta_0. \quad (14)$$

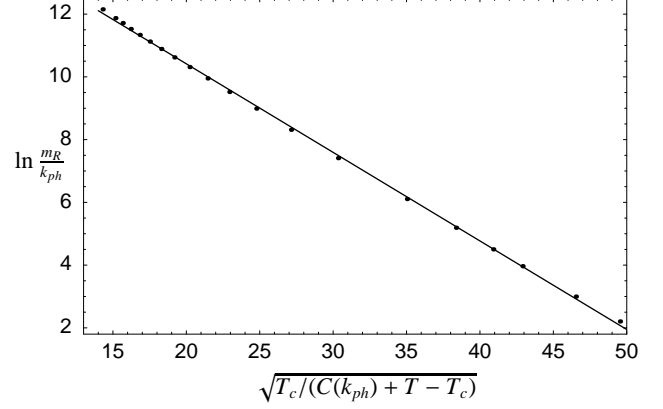


FIG. 2: Essential scaling for  $T > T_c$  with  $C(k_{ph})/t = 10^{-5}$ . The dots are the numerical solution to the flow equation.

This yields for the flow of the rescaled effective potential, now at fixed  $\tilde{\delta}$  instead of fixed  $\delta$ ,

$$k\partial_k u_k = -2u_k + \eta(k)\tilde{\delta} \frac{\partial u_k}{\partial \tilde{\delta}} + \frac{t^2}{Tk^2} k\partial_k U_k|_{\tilde{\delta}}, \quad (15)$$

implying for the renormalized field expectation value  $\kappa$  and bosonic quartic coupling  $\lambda_\phi$

$$\partial_k \kappa = -\frac{1}{\lambda_\phi} \frac{\partial}{\partial \tilde{\delta}} \partial_k u_k|_{\tilde{\delta}=\kappa}, \quad \partial_k \lambda_\phi = \frac{\partial^2}{\partial \tilde{\delta}^2} \partial_k u_k|_{\tilde{\delta}=\kappa}. \quad (16)$$

The bosonic contribution to the anomalous dimension in SSB is the same as for the linear  $O(2)$  model in two dimensions

$$\eta(k) = \frac{1}{\pi} \frac{\lambda_\phi^2 \kappa}{(1 + 2\lambda_\phi \kappa)^2} = \frac{1}{4\pi\kappa} + O(\kappa^{-2}), \quad (17)$$

while the fermionic contribution can be neglected.

The  $\beta$ -functions can be expanded in powers of  $(2\lambda_\phi \kappa)^{-1}$ . In particular, to leading order one finds

$$k\partial_k \kappa = \beta_\kappa = O(\kappa^{-1}), \quad k\partial_k \lambda_\phi = -2\lambda_\phi + \frac{\lambda_\phi^2}{2\pi} + O(\kappa^{-1}), \quad (18)$$

resulting in an infrared attractive fixed point for  $\lambda_\phi$ . The running of  $\kappa$  can be mapped to the running of the coupling in the non-linear  $\sigma$ -model for arbitrary  $O(N)$ -models [6, 30, 31]. For  $N = 2$  also the term  $\sim \kappa^{-1}$  in  $\beta_\kappa$  has to vanish, even though this is not seen in the truncation (3). The flow of the classical linear  $O(2)$ -model has already been studied with much more extended truncations [31], taking into account an arbitrary  $\tilde{\delta}$ -dependence of  $u(\tilde{\delta})$  and arbitrary  $\tilde{\delta}$ -dependent wave function renormalizations, e.g.  $A(\tilde{\delta})$ . All these functions flow quickly to a scaling form which depends only on one parameter  $\kappa$ . Evaluated for the scaling solution one finds to a very good approximation for  $\kappa > \kappa_{KT}$

$$k\partial_k \kappa = \beta_\kappa = \alpha |\kappa_* - \kappa|^{\frac{3}{2}} \Theta(\kappa_* - \kappa) \quad (19)$$

with  $\kappa_* = 0.248$  and  $\alpha = 2.54$  [31]. We will use this improved truncation for  $\beta_\kappa$  for  $k < T/12$  for the region  $\kappa > \kappa_{KT} = 0.1$ .

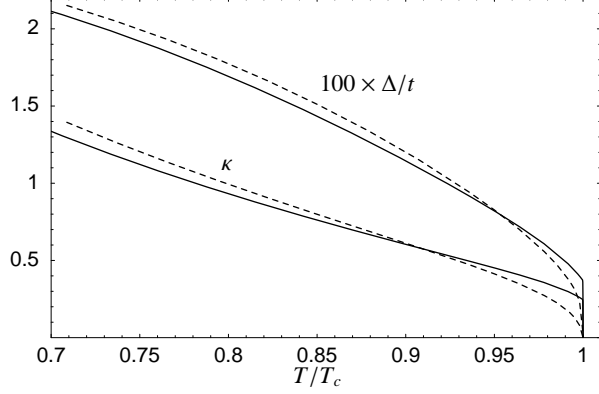


FIG. 3: Temperature dependence of d-wave superconducting order. We plot the renormalized superfluid density  $\kappa$  and the fermionic gap  $\Delta(0, \frac{\pi}{2})/t$ . Solid/dashed lines are based on eq. (19)/(3).

This results in the solid lines in fig. 1, whereas the truncation (3) gives the dashed lines.

We define the critical temperature  $T_c$  as the temperature for which the field expectation value  $\kappa$  just vanishes at the physical scale  $k_{ph}/t = 10^{-9}$ , see fig. 1. For  $T < T_c$  one finds a superfluid condensate in a probe of macroscopic size of roughly 1 cm. The dependence of  $T_c$  on the specific choice of  $k_{ph}$  is extremely weak. For our parameters  $t = 0.3$  eV corresponds to  $T_c = 128$  K while for alternative  $\lambda_d/t = 1/2, 1$  we find  $T_c = 58.3, 366$  K, respectively. We also note that  $T_c$  is only mildly affected by the improvement of the truncation (19).

A specific feature of the KT phase transition is essential scaling for the temperature dependence of the correlation length  $\xi = m_R^{-1} = (\bar{m}^2/A)^{-\frac{1}{2}}$  just above  $T_c$ ,

$$m_R = ce^{-\frac{b}{\sqrt{C(k_{ph})+T-T_c}}} \quad (20)$$

The finite size correction  $C(k_{ph}) \sim (1 + \frac{a}{2}\kappa_*^{\frac{1}{2}} \ln \frac{\Lambda}{k_{ph}})^{-2}$  vanishes logarithmically for  $k_{ph} \rightarrow 0$ . This follows from eq. (19). Close to the critical temperature the omission of the finite size correction (i.e.  $C(k_{ph}) = 0$ ) yields to a substantial derivation from the straight line in fig. 2.

For  $T > T_c$  one finds that  $\kappa(k)$  reaches zero at a scale  $k_{SR} > k_{ph}$ . Continuing the flow in the symmetric regime for  $k_{SR} > k > k_{ph}$  yields  $m_R = 0.50 k_{SR}$ . We show  $m_R(T)$  in fig. 2 (same  $t', \lambda_d, \mu$  in all figs.) and find that essential scaling is smoothened due to  $k_{ph} > 0$ .

From fig. 1 we can also extrapolate the pseudo-critical temperature  $T_{pc}$ . This corresponds to the temperature where  $\kappa(k)$  bends over immediately after becoming non-zero at  $k = k_{SSB}$  and is driven to zero again more or less at the same scale  $k_{SR} \approx k_{SSB}$ . In other words,  $T_{pc}$  corresponds to the highest temperature for which the initial flow in the symmetric regime hits a vanishing mass term  $\bar{m}^2$  and therefore a divergent effective four fermion vertex  $\sim \bar{m}^{-2}$ . Often this temperature is associated with the critical temperature. We find  $T_{pc}/t = 0.0372$ , only slightly above the value of the critical temperature  $T_c/t = 0.0368$ .

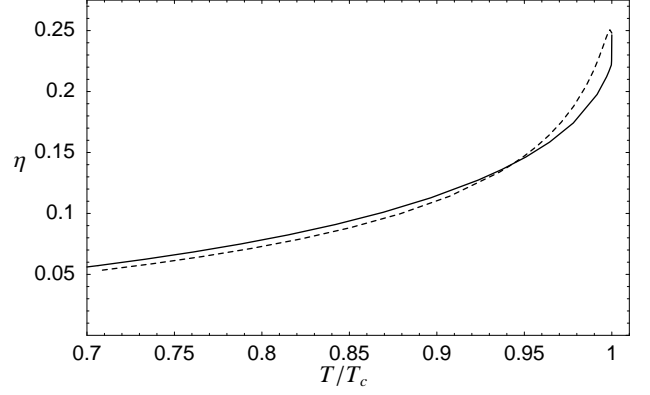


FIG. 4: Temperature dependence of the critical exponent  $\eta$ . The solid line is based on eq. (19), the dashed line on eq. (3).

It is characteristic for two dimensions that the renormalized order parameter  $\kappa$  jumps at  $T_c$  even though the transition shows scaling behavior. (Strictly speaking, this discontinuity only occurs for  $k_{ph} \rightarrow 0$  whereas it is smoothened for nonzero  $k_{ph}$ .) The scaling solution at  $T_c$  corresponds to a fixed point  $\kappa_*$  where  $\beta_\kappa(\kappa_*) = 0$  with  $\beta_\kappa(\kappa < \kappa_*) > 0$ . In the region  $\kappa(k) < \kappa_*$  the flow drives  $\kappa$  to zero and the system is in the symmetric phase. In contrast, for  $\kappa > \kappa_*$  the flow can never cross the fixed point. Therefor  $\kappa(k_{ph})$  remains larger than  $\kappa_*$ , resulting in a minimal value for  $\kappa$  in the ordered phase. For  $k_{ph} \rightarrow 0$  this results in a jump  $\Delta\kappa = \kappa_*$  between the two phases. In superfluids  $\kappa(k_{ph})$  determines the renormalized superfluid density. The value resulting in the vortex picture [5] is  $\kappa_* = \frac{1}{\pi}$ . Our truncation (19) is already parameterized in terms of  $\kappa_*$ . In fig. 3 we plot the value of the superfluid density  $\kappa(k_{ph})$  as well as the gap in the electron propagator  $\Delta(0, \frac{\pi}{2})$  in a fixed momentum direction. In the present truncation  $\Delta$  would vanish for  $k_{ph} \rightarrow 0$ , whereas  $\kappa$  would remain nonzero. In order to settle this issue more precisely one would have to follow the flow of the Yukawa coupling that we neglected here. For  $t = 0.3$  eV and  $T = T_c/2$  we obtain  $\Delta(0, \frac{\pi}{2}) = 7.7$  meV.

We finally turn to the anomalous dimension for  $T \leq T_c$ . For weakly inhomogeneous fields (low  $\mathbf{q}^2$ ) the gradient terms in the effective theory (Landau theory) for the composite bosons  $\phi$  take the form

$$\Gamma = \frac{1}{T} \int_q \phi^*(q) A(\mathbf{q}^2, k_{ph}) \mathbf{q}^2 \phi(q). \quad (21)$$

Our truncation (8) has not yet taken the  $\mathbf{q}^2$ -dependence of  $A$  into account: the flow equations apply for  $A(k) = A(\mathbf{q}^2 = 0, k)$ . Nevertheless, the momentum dependence of  $A$  can be easily inferred by noting that  $\mathbf{q}^2$  also acts as an infrared cutoff such that  $R_k$  becomes ineffective for  $k^2 < \mathbf{q}^2$ . A reasonable approximation uses  $\partial_k A(\mathbf{q}^2, k) = \partial_k A(\mathbf{q}^2 = 0, k) \Theta(k^2 - \mathbf{q}^2)$ . The solution at  $k_{ph}$  reads for  $\mathbf{q}^2 > k_{ph}^2, q = \sqrt{\mathbf{q}^2}$ ,

$$A(q^2) = \bar{A}(q/\bar{q})^{-\eta}, \quad \eta = \int_{\ln \bar{q}}^{\ln \bar{q}} d \ln k \eta(k) / \ln(\bar{q}/q). \quad (22)$$

In fig. 4 we show  $\eta$  for  $\bar{q}/q = e^5$  and observe the characteris-

tic dependence on  $T$ . As a result the static correlation function  $\langle\phi(\mathbf{r})\phi^*(0)\rangle_c$  decreases  $\sim r^{-\eta}$  for large  $r$ . For a superconductor one replaces in eq. (21)  $\mathbf{q}^2$  by  $-(\nabla - 2ie|\mathbf{A}|)^2$  with  $\mathbf{A}$  the space components of the electromagnetic potential. Inserting  $\phi = \phi_0$  this results in an anomalous Landau theory for the magnetic field  $B = \partial_1 A_2 - \partial_2 A_1$ . Instead of the photon mass

term  $\sim \mathbf{A}^2$  responsible for ordinary superconductivity in three dimensions we now find a term  $|\mathbf{A}|^{2-\eta}$ . This may open a window for a measurement of the anomalous dimension.

Acknowledgment: H. C. Krah1 thanks P. Strack and S. Diehl for useful discussions and comments.

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